UNIVERSAL ABELIAN COVERS OF SUPERISOLATED SINGULARITIES

JAN STEVENS

ABSTRACT. We give explicit examples of Gorenstein surface singularities with integral homology sphere link, which are not complete intersections. Their existence was shown by Luengo-Velasco, Melle-Hernández and Némethi, thereby providing counterexamples to the universal abelian covering conjecture of Neumann and Wahl.

The topology of a normal surface singularity does not determine the analytical invariants of its equisingularity class. Recent partial results indicated that this nevertheless could be true under two restrictions, a topological one, that the link of the singularity is a rational homology sphere, and an analytical one, that the singularity is $\mathbb{Q}$-Gorenstein. Neumann and Wahl conjectured that the singularity is then an abelian quotient of a complete intersection singularity, whose equations are determined in a simple way from the resolution graph [14]. Counterexamples were found by Luengo-Velasco, Melle-Hernández and Némethi [7], but they did not compute the universal abelian cover of the singularities in question. The purpose of this paper is to provide explicit examples.

We give examples both of universal abelian covers and of Gorenstein singularities with integral homology sphere link, which are not complete intersections. We mention here one example with both properties, see Propositions 3 and 7.

Proposition. *The Gorenstein singularity in $(\mathbb{C}^6, 0)$ with ideal generated by the maximal minors of the matrix*

$$\begin{pmatrix} u & y-z & 8s^2-9y-9z & w \\ 5s^2-9y-9z & v & w & u^2-46s^3+54ys+54zs+us^2 \end{pmatrix}$$

and three further polynomials

$$8u^2s + (8s^2-9y-9z)(4s^2-9y-9z) + 27(y-z)^2 \,,$$
$$8(5s^2-9y-9z)us + w(4s^2-9y-9z) + 27(y-z)v \,,$$
$$8(5s^2-9y-9z)^2s + (u^2-46s^3+54ys+54zs+us^2)(4s^2-9y-9z) + 27v^2 \,,$$

has integral homology sphere link. The exceptional divisor on the minimal resolution is a three-cuspidal rational curve with self-intersection -1.
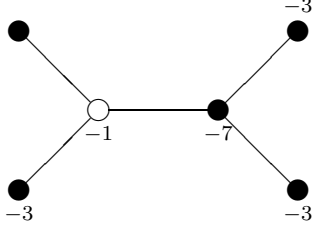*The singularity is the universal abelian cover of a hypersurface singularity with equation*

$$27t^2 + (4s^3+u^2)^3 + 2u^5s^2 - 20u^3s^5 - 4us^8 + u^4s^4 = 0 \,,$$

which is the suspension of an irreducible plane curve singularity with Puiseux pairs $(3, 2)$ *and* $(10, 3)$. *The link has as first homology group* $\mathbb{Z}_3$ *and the resolution graph is*

This graph occurs in [14, Examples 2] as an example violating the so-called semigroup condition. It is not the graph of a superisolated singularity, nor is the universal abelian cover of the type which can occur as cover of such a singularity. But our method of computing for superisolated singularities also yields the stated equations.

A surface singularity $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is called superisolated if blowing up the origin once resolves the singularity. Writing the equation $f = f_d + f_{d+1} + \cdots$ in its homogeneous parts the condition becomes that f_d defines a plane curve with isolated singularities, through which the curve $f_{d+1} = 0$ does not pass. These singularities were introduced by Luengo to give counterexamples to the smoothness of the μ-const stratum [6]. They provide a class of singularities lying outside the usual examples (they are degenerate for their Newton diagram), yet they are easy to handle because most computations reduce to the plane curve $f_d = 0$ alone. To obtain a singularity with rational homology sphere link from an irreducible curve one needs that it is also locally irreducible, so it is a rational curve with only (higher) cusp singularities. The universal abelian cover of such a superisolated singularity is a cyclic cover of degree d.

The typical example of a superisolated singularity has the form $f_d + l^{d+1}$, where l is a general linear function. This suggests to generalise the problem and study coverings of the whole Yomdin series $f_d + l^{d+k}$. In general these singularities are not quasi-homogeneous, which complicates the computations. But the two limiting cases of the series are homogeneous: for $k = \infty$ we have the non-isolated homogeneous singularity f_d, while $k = 0$ gives the cone over a smooth plane curve of degree d. To find a cyclic cover of degree d we note that in both cases we have an irreducible curve C, embedded by a very ample line bundle H of degree d. The affine cone over C has local ring $\bigoplus H^0(C, nH)$. Let L be a line bundle with $dL = H$. The ring $\bigoplus H^0(C, nL)$ is the local ring of a quasi-homogeneous singularity (the quasi-cone $X(C, L)$), which is a d-fold cover of the cone over C. For a smooth curve there are d^{2g} line bundles L satisfying $dL = H$, but for singular curves the number is less. In particular, in the case of interest to us, where C is a rational cuspidal curve, the Jacobian of C is a unipotent group and there is a unique line bundle L satisfying $dL = H$. For this line bundle we determine the ring $\bigoplus H^0(C, nL)$. We find in this way the d-fold cover of the singularity f_d. This is again a non-isolated singularity. If we understand its equations well enough, we can write down a series of singularities, and quotients of suitable elements are coverings of singularities in the series of f_d. It is also worthwhile to look at other elements of the series; in this way the equations in the proposition above were obtained. We succeed for the case $d = 4$. But for a number of curves of higher arithmetical genus, where we have determined the structure of the ring $\bigoplus H^0(C, nL)$, the equations of the non-isolated singularity are complicated and have no apparent structure. Therefore we do not write them, nor do we give a series of singularities.

Superisolated singularities with $d = 5$ give counterexamples [7] to the conjectures by Némethi and Nicolaescu [10] on the geometric genus of Gorenstein singularities with rational homology sphere link. We have made a detailed study of this case. If the splice quotient of Neumann and Wahl exists, and that is the case for an exceptional curve with at most two cusps, it has the predicted geometric genus.

In the first section we discuss the relation between topological and analytical invariants. The next section recalls the conjectures of Némethi and Nicolaescu, and of Neumann and Wahl. The third section contains generalities on superisolated singularities, while the next one presents the results for the case $d = 4$. The section on degree five focuses on the geometric genus. The next section treats generalisations. In the last section we discuss the conjectures in the light of our examples.

The computer computations for this paper were done with the computer algebra system SINGULAR [3]. Thanks are due to the referee for his detailed comments and suggestions.

1. ANALYTICAL AND TOPOLOGICAL INVARIANTS

The conjectures on $\mathbb{Q}$-Gorenstein surface singularities with rational homology sphere link fit in the wider framework of the following question.

Question. *Which discrete data are needed to know a normal surface singularity?*

One interpretation of 'knowing a singularity' is that we can write down equations. As we only have discrete data, such equations necessarily describe a family of equisingular surfaces. At the very least one should know the geometric genus of the singularity, the most basic analytical invariant, which is constant in equisingular families (this requirement excludes the multiplicity). The first thing needed is the topology of the singularity, or what amounts to the same, the resolution graph.

Let $\pi\colon (\widetilde{X}, E) \to (X, p)$ be any resolution of a normal surface singularity (X, p) with exceptional set $E = \bigcup_{i=1}^{r} E_i$. The intersection form $E_i \cdot E_j$ is a negative definite quadratic form. It can be coded by a weighted graph, with vertices v_i corresponding to the irreducible components E_i of the exceptional set with weight $-b_i = E_i \cdot E_i$; two vertices are joined by an edge, if the corresponding components intersect, with edge weight the intersection number. The numerical class K of the canonical bundle of M provides a characteristic vector: by the adjunction formula we have $E_i \cdot (E_i + K) = 2p_a(E_i) - 2$. The resolution graph of π is the weighted graph obtained by giving the vertices as additional weight the arithmetical genus of the corresponding component, written in square brackets. It encodes all necessary information for the calculus of cycles on E, but it carries no information on the singularities of E. Therefore one usually requires that all irreducible components are smooth and that they intersect transversally; this is called a good resolution. All edge weights are then equal to 1, and are not written. According to custom we also suppress the vertex weights [0] and -2. By the resolution graph of a singularity we mean the graph of the minimal good resolution.

For rational singularities the resolution graph suffices to 'know' the singularity, by a celebrated result of Artin. By Laufer and Reid the same holds for minimally elliptic singularities. Beyond these classes of singularities this is no longer true, as shown by the following example of Laufer, see [19, Example 6.3]; it occurs on several places in [9], starting with 2.23.

1.1. **Example.** Consider the resolution graph:
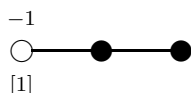
The singularity is numerically Gorenstein, with $K = -3E_0 - 2E_1 - E_2$ on the minimal resolution, where E_0 is the elliptic curve, which is intersected by the rational curve E_1 in the point P, and E_2 is the remaining rational curve. Let $\mathcal{O}_{E_0}(-Q)$ be the normal bundle of E_0 on the minimal resolution. One has $E_0 \cdot (E_0 + K) = 2Q - 2P$, as divisor on E_0. So by adjunction $2P = 2Q$, if the singularity is Gorenstein.

To compute $p_g = H^1(\mathcal{O})$ one can use computation sequences for S.S.-T. Yau's elliptic sequence; a convenient reference is [9]. Starting from $Z = E_0 + E_1 + E_2$ one considers in

order the cycles $Z + E_0$, $Z_1 = Z + E_0 + E_1$ and $Z_2 = -K = Z_1 + E_0$. One looks at the long exact cohomology sequences of the short exact sequences connecting these cycles. The most relevant ones are

$$0 \longrightarrow \mathcal{O}(-Z - E_0) \longrightarrow \mathcal{O}(-Z) \longrightarrow \mathcal{O}_{E_0}(Q - P) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-Z_1 - E_0) \longrightarrow \mathcal{O}(-Z_1) \longrightarrow \mathcal{O}_{E_0}(2Q - 2P) \longrightarrow 0 \, .$$

As $H^1(\mathcal{O}(-Z_1 - E_0)) = H^1(\mathcal{O}(K)) = 0$ one finds $p_g \leq 3$. Moreover, if $2P \neq 2Q$, the singularity is not Gorenstein and $p_g = 1$. If $2P = 2Q$, then $p_g \geq 2$, and $p_g = 2$ if $P \neq Q$. If $P = Q$ both cases $p_g = 2$ and $p_g = 3$ are possible.

In the following we give equations for all cases. For double points $z^2 = f(x, y)$ this can be done starting from the topological type of the branch curve, but in general it is very difficult to find equations. The easiest approach is to first compute generators of the local ring. This works for quasi-homogeneous singularities, where one can use Pinkham's method [19]. By computing deformations of positive weight one finds other singularities. As the graph is star-shaped, there exist quasi-homogeneous singularities with this graph. Their coordinate ring is the graded ring $R = \bigoplus H^0(\mathcal{O}_{E_0}(nQ - \lceil \frac{2}{3}n \rceil P))$. We may take the point Q as origin of the group law on the elliptic curve. The structure of the ring R depends on the order of the point P on the elliptic curve.

1.1.1. $\boldsymbol{P = Q}$, $\boldsymbol{p_g = 3}$.
In this case the singularity is Gorenstein by a result of S.S.-T. Yau, see [9, 3.5], and all singularities are deformations of the quasi-homogeneous ones. The weighted homogeneous ring R has three generators, in degrees 1, 6 and 9. The whole equisingular stratum is given by

$$x^2 + y^3 + a_0 y z^{12} + z^{18} + a_{-1} y z^{13} + a_{-2} y z^{14} + a_{-3} y z^{15} + a_{-4} y z^{16} \, .$$

1.1.2. $\boldsymbol{P = Q}$, $\boldsymbol{p_g = 2}$.
A singularity of this type is still a deformation of a quasi-homogeneous singularity, but now a nonnormal one. The resolution specialises to the resolution of the quasi-homogeneous singularity and blowing down the exceptional set of the total space yields a family specialising to a nonnormal singularity with $\delta = l(\widetilde{\mathcal{O}}/\mathcal{O}) = 1$, where $\widetilde{\mathcal{O}}$ is the normalisation of the local ring. An example of such a singularity is provided by the subring of $\mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^{18})$ generated by x, y, xz, yz, z^2 and z^3. In computing a deformation of positive weight of this singularity there is no guarantee that the computation stops, but with some luck we succeeded. For the deformed singularity two variables can be eliminated and the result of our computation is a determinantal singularity in $\mathbb{C}^4$, for which we give the equisingular stratum. This will contain most singularities of this type, but maybe not all. In new variables (x, y, z, w) we get the equations

$$\text{Rank} \begin{pmatrix} 2w - xz & 4y + x^2 & z \\ 4xy - z^2 - 4z^4 + b_0 x^3 z^2 + (a_{-1}xz + a_0 x^2)(z^2 - x^3) & 2w + xz & y + x^2 \end{pmatrix} \leq 1 \, .$$

Checking that this singularity has indeed the required resolution graph can be done by embedded resolution, for which point blow-ups suffice. Eliminating w, i.e., projecting onto the first three coordinates, gives the non-isolated hypersurface singularity

$$z^2(z^2 - 2x^3 - 6xy + 4z^4 - b_0 x^3 z^2 + (a_{-1}xz + a_0 x^2)(x^3 - z^2)) + (4y + x^2)(y + x^2)^2 \, .$$

The singular locus is given by $z = y + x^2 = 0$. Let ω be the generator of the dualising sheaf. Then $z\omega$ and $(y + x^2)\omega$ lift to regular differential forms on the normalisation. So indeed $p_g = 2$.

This example is related to examples of Okuma [17, 6.3].

1.1.3. $P \neq Q$, *but* $2P = 2Q$. Every such singularity is a deformation of a quasi-homogenous one. Generators of the weighted homogeneous ring have degree 2, 3 and 7. The equisingular stratum is

$$x^2 + z(y^4 + a_0 y^2 z^3 + z^6) + a_{-1} y z^6 + a_{-2} y^2 z^5 + a_{-4} y^2 z^6 \; .$$

1.1.4. $2P \neq 2Q$. Here there are two moduli, one being the modulus of the elliptic curve, the other the class of the divisor $P - Q$ in the Jacobian of the curve. We start again from the quasi-homogeneous case. The dimension of the group $H^0(\mathcal{O}_{E_0}(nQ - \lceil \frac{2}{3}n \rceil P))$ is $\lfloor \frac{n}{3} \rfloor$. We consider the plane cubic

$$(1) \qquad \xi^2 \eta + \eta^2 \xi + \zeta^3 + \lambda \xi \eta \zeta$$

with inflexion point $Q = (1 : 0 : 0)$ and inflexional tangent $\eta = 0$. A section of $H^0(\mathcal{O}_{E_0}(3Q - 2P))$ can be given by the rational function $\frac{l}{\eta}$, where l is the equation of the tangent line in the point P. For $H^0(\mathcal{O}_{E_0}(4Q - 3P))$ we take the function $\frac{q}{\eta^2}$, where q is a conic intersecting the the curve in Q with multiplicity 2 and in P with multiplicity 3. Continuing in this way one gets generators of the local ring. We refrain from giving complicated general formulas. The formulas simplify if $3P = 3Q$, but on the other hand the embedding dimension increases by one. Following the principle of first describing the most special singularity we take for P the inflexion point $(0 : 1 : 0)$. The function $\frac{\xi}{\eta}$ is a section of $H^0(\mathcal{O}_{E_0}(3Q - 2P))$ and $H^0(\mathcal{O}_{E_0}(4Q - 3P))$. We homogenise and write the sections as forms of degree k. With this convention we find generators $x = \xi \eta^2$, $y = \xi \eta^3$, $z = \zeta \xi \eta^3$, $u = \zeta \xi \eta^4$, $v = \zeta^2 \xi \eta^4$ and $w = \zeta^3 \xi \eta^5$. We need a generator in degree 9, as $xu = yz$. There are 10 equations, which can be written in *rolling factors format* [20, 23]. There are two deformations of positive weight, which respect this format; in fact, they can be written as deformation of the equation (1):

$$(2) \qquad \xi^2 \eta + \eta^2 \xi + \zeta^3 + \lambda \xi \eta \zeta + a_{-1} \xi^2 \eta^2 + a_{-3} \xi^2 \eta^3 \zeta \; .$$

The equations consist of the six equations of a scroll:

$$\text{Rank} \begin{pmatrix} x & z & v & y \\ z & v & w & u \end{pmatrix} \leq 1$$

and four additional equations, obtained by multiplying the equation (2) with suitable factors. The transition from one equation to the next involves the replacement of one occurring entry of the top row of the matrix by the one standing below it. We get

$$y^3 + x^4 + xw + \lambda xyz + a_{-1} yx^3 + a_{-3} ux^3 \; ,$$
$$uy^2 + zx^3 + zw + \lambda yz^2 + a_{-1} yx^2 z + a_{-3} ux^2 z \; ,$$
$$u^2 y + vx^3 + vw + \lambda yzv + a_{-1} yx^2 v + a_{-3} ux^2 v \; ,$$
$$u^3 + wx^3 + w^2 + \lambda yzw + a_{-1} yx^2 w + a_{-3} ux^2 w \; .$$

Furthermore there is a deformation of degree 0, corresponding to moving the point P. It does not respect the rolling factors format; as infinitesimal deformation we change the equation of degree 9 to $ux - yz + \varepsilon w$, showing again that the embedding dimension drops for $3P \neq 3Q$. We did not succeed in computing the higher order terms, which seem to involve power series. Only for the special value $\lambda = 0$ of the modulus the computation ended after a finite number

of steps. We give the result.

$$\text{Rank} \begin{pmatrix} z & v & y \\ v & w & u - \varepsilon x^2 \end{pmatrix} \le 1 \ ,$$

$$(u - \varepsilon^4 x^2)x - zy + \varepsilon w \ ,$$

$$z^2 - vx + \varepsilon(u - \varepsilon^4 x^2)y - 2\varepsilon^3 vx \ ,$$

$$vz - wx + \varepsilon(u - \varepsilon x^2)(u - \varepsilon^4 x^2) - 2\varepsilon^3 wx \ ,$$

$$y^3 + wx + x^4 - \varepsilon u^2 \ ,$$

$$wz + (u - \varepsilon^4 x^2)y^2 + zx^3 - 2\varepsilon^2 vyx - z\varepsilon^3 x^3 \ ,$$

$$wv + (u - \varepsilon x^2)(u - \varepsilon^4 x^2)y + vx^3 - 2\varepsilon^2 wyx - v\varepsilon^3 x^3 \ ,$$

$$w^2 + (u - \varepsilon x^2)^2(u - \varepsilon^4 x^2) + wx^3 - 2\varepsilon^2 wx(u - \varepsilon x^2) - w\varepsilon^3 x^3 \ .$$

Similar examples exist with E_0 replaced by other minimally elliptic cycles, such as a chain of rational curves. But if E_0 is a rational cuspidal curve, there are no torsion points, as the group structure of a cuspidal cubic is that of the additive group $\mathbb{G}_a$, and there is only one possibility for a Gorenstein singularity, the one with $p_g = 3$, giving the singularity E_{36} (in Arnold's notation); a quasihomogeneous equation is $x^2 + y^3 + z^{19}$.

2. The conjectures

The conjecture has been made that the geometric genus is topological, if one assumes a topological restriction, that the link of the singularity is a rational homology sphere, and an analytical one, that the singularity is $\mathbb{Q}$-Gorenstein. In fact, Némethi and Nicolaescu proposed a precise formula:

$$p_g = \mathbf{sw}(M) - (K^2 + r)/8 \ ,$$

where $\mathbf{SW}(M)$ is the Seiberg-Witten invariant of the link M, as before K is the canonical cycle and r the number of components of the exceptional divisor; for details see [10].

For smoothable Gorenstein singularities this conjecture can be formulated in terms of the signature of the Milnor fibre. We recall Laufer's formula

$$\mu = 12p_g - b_1(E) + b_2(E) + K^2 \ .$$

Together with $\mu_0 + \mu_+ = 2p_g$ and $\mu_0 = b_1(E)$ we find Durfee's signature formula

$$\sigma = 8p_g - b_2(E) - K^2 \ .$$

So the conjecture is $\mathbf{SW}(M) = -\sigma/8$. This conjecture generalises the Casson Invariant Conjecture of Neumann and Wahl [13], that for an complete intersection with integral homology sphere link $\sigma/8$ equals the Casson invariant.

As an outgrowth of their work Neumann and Wahl even came up with a way to write down equations from the resolution graph. What is known about it, is very well described in Wahl's survey [24]. In the case of a rational homology sphere link satisfying certain conditions one gets equations together with an action of the finite group $H = H_1(M, \mathbb{Z})$. Given a singularity (X, p) with H finite, the maximal unramified abelian cover of $X \setminus \{p\}$ is a Galois cover with covering transformation group H, which can be completed with one point. We get a map $(\widetilde{X}, p) \to (X, p)$, which is called the universal abelian cover of (X, p). The recipe for the equations involve the splice diagram of the singularity, see [14, 15]. Here we recall only the case of a resolution graph of a quasi-homogeneous singularity, already given in [12]. There is one central curve with k arms, which are resolution graphs of cyclic quotient singularities with a group of order α_i, $i = 1, \ldots, k$. The associated $k - 2$ equations define a Brieskorn-Pham complete intersection: one has linear equations in $x_i^{\alpha_i}$ with general enough coefficient matrix.

The group H acts diagonally. In general these equations are the building blocks, which have to be spliced together, leading to splice type equations. This requires conditions on the graph, which are called the 'semigroup condition' (in order to be able to splice) and the 'congruence condition' (to have an action of H). The resulting singularity is said to be a splice quotient.

3. SUPERISOLATED SINGULARITIES

A surface singularity $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is called superisolated if blowing up the origin once resolves the singularity. We investigate what this means in terms of equations in a more general situation.

Proposition 1. *Let $f = f_d + f_{d+k} + \cdots$ be the decomposition in homogeneous parts. The first blow up has only isolated singularities of suspension type, of the form $z^k = g_i(x, y)$, where the $g_i(x, y)$ are the local equations of the singularities of the exceptional curve (in $\mathbb{P}^2$), if and only if the homogeneous polynomial f_d defines a plane curve with isolated singularities, through which the curve $f_{d+k} = 0$ does not pass.*

In this situation the resolution graph of the singularity depends only on d, k and the topological type of the singularities g_i.

Singularities satisfying the conditions of the proposition are said to be of *Yomdin type* [1], as the most important example is a Yomdin series $f_d + l^{d+k}$, where l is a general linear form.

Proof. We compute the blow-up. We may assume that $(0 : 0 : 1)$ is a singular point of the plane curve f_d and compute in the chart with coordinates (ξ, η, z), where $(x, y, z) = (\xi z, \eta z, z)$. Then the strict transform of f is

$$\bar{f}(\xi, \eta, z) = f_d(\xi, \eta, 1) + z^k(f_{d+k}(\xi, \eta, 1) + z f_{d+k+1}(\xi, \eta, 1) + \dots).$$

The origin is an isolated singularity of $\bar{f}$ if and only if $(0 : 0 : 1)$ is an isolated singularity of f_d. This singularity is a k-fold suspension singularity if and only if $f_{d+k}(\xi, \eta, 1) + z f_{d+k+1}(\xi, \eta, 1) + \dots$ is a unit, so $f_{d+k}(0, 0, 1) \neq 0$. To determine the embedded resolution it suffices to consider the singularity $f_d(\xi, \eta, 1) + z^k$. The other singular points of f_d can be treated in a similar way. $\square$

In particular, for a superisolated singularity the first blow up gives the minimal resolution. It has the curve $C\colon f_d = 0$ as exceptional divisor with self intersection $-d$. In fact, the normal bundle of C is $\mathcal{O}_C(-1)$, where $H = \mathcal{O}_C(1)$ is the hyperplane bundle of the embedding of C in $\mathbb{P}^2$. The link M of the singularity is a rational homology sphere if C is homeomorphic to S^2 (we restrict ourselves to the case of irreducible curves). This implies that it is locally irreducible, so C is an irreducible rational curve with only (higher) cusp singularities. Then $H_1(M, \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, generated by a small loop around the exceptional curve. The universal abelian cover of such a superisolated singularity is a cyclic cover of degree d. We can describe it in the following way [16]. Consider the $\mathbb{Q}$-divisor $\frac{1}{d}C$ on the minimal resolution $\pi\colon \widetilde{X} \to X$ of the singularity. There exists a divisor D on $\widetilde{X}$ such that C is linearly equivalent to dD. Then $\pi_* D$ is a $\mathbb{Q}$-Cartier divisor of index d on X. The local ring of the covering Y, as $\mathcal{O}_X$-module, is $\bigoplus_{j=0}^{d-1} \mathcal{O}_X(-j\pi_* D) = \bigoplus_{j=0}^{d-1} \pi_* \mathcal{O}_{\widetilde{X}}(-jD)$. The sheaf $\bigoplus_{j=0}^{d-1} \mathcal{O}_{\widetilde{X}}(-jD)$ defines a d-fold cover of $\widetilde{X}$, branched along C, which is a partial resolution $\overline{Y}$ of Y. It has C as exceptional divisor, this time with self intersection -1, and singularities at the singular points of C, of suspension type $z^d = g_i(x, y)$. It is now easy to find the resolution graph of the universal abelian cover, as already noted in [7, 3.2]. But is is very hard to find D and to determine the ring structure explicitly. We therefore compute on the curve C.

The exceptional curve C on the partial resolution $\overline{Y}$ has as normal bundle the dual of a dth root of $H = \mathcal{O}_C(1)$. For a rational cuspidal curve this dth root is unique. In general,

for a singular curve C with normalisation $n\colon \widetilde{C} \to C$, there is an exact sequence [4, Exercise II.6.8]

$$0 \longrightarrow \bigoplus_{P \in C} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \longrightarrow \operatorname{Pic} C \longrightarrow \operatorname{Pic} \widetilde{C} \longrightarrow 0 \ ,$$

where $\widetilde{\mathcal{O}}_P$ is the normalisation of the local ring of P, and the star denotes the group of units in a ring. For a singularity with one branch $\widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^*$ is a unipotent algebraic group. With $\operatorname{Pic} \widetilde{C} = \mathbb{Z}$ for a rational curve this gives the uniqueness.

4. Degree four

In this section we construct examples of universal abelian covers, which are not complete intersections. To this end we consider superisolated singularities with lowest degree part of degree four. The singularities on cuspidal rational quartic curves can be E_6, A_6, $A_4 + A_2$ or $3A_2$. This list is easily obtained, as the sum of the Milnor numbers has to be 6. Equations can be found in [21].

4.1. **Rational curves with an E_6-singularity.** There are two projectively inequivalent quartic curves with an E_6-singularity. One admits a $\mathbb{C}^*$-action, and can be given by the equation $x^4 - y^3 z = 0$. The other type can be written as $(x^2 - y^2)^2 - y^3 z = 0$. In the first case the line $z = 0$ is a hyperflex, while it is an ordinary bitangent in the second case.

The first type gives rise to a series of quasi-homogeneous isolated surface singularities

$$f_k\colon \ x^4 - y^3 z - z^{4+k} \ .$$

In particular, for $k = 1$ we have a superisolated singularity, whose minimal resolution has the given rational curve as exceptional divisor with self-intersection -4. The minimal good resolution has dual graph:
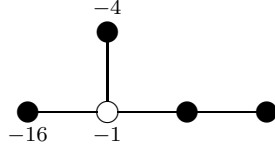
A splice type equation for this graph is $\xi^4 - \eta^3 - \zeta^{16}$. The action of $H_1(M, \mathbb{Z})$ is $\frac{1}{4}(3, 4, 1)$. This means that we have a diagonal action of $\mathbb{Z}_4$ with generator $(\xi, \eta, \zeta) \mapsto (-i\xi, \eta, i\zeta)$. Invariants are $x = \xi\zeta$, $y = \eta$ and $z = \zeta^4$ (note that $\xi^4 = \eta^3 + \zeta^{16}$) and the quotient is exactly the quasi-homogeneous superisolated singularity. In fact, the same group acts on the singularities $\xi^4 - \eta^3 - \zeta^{12+4k}$ with quotient exactly f_k, but in general this will not be the universal abelian cover. The link need not even be a rational homology sphere (an example in case is $k = 12$).

For $k = 2$ the graph is:
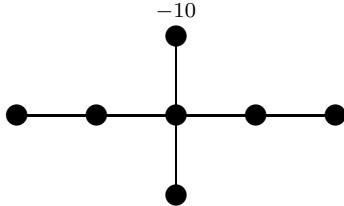
The quasi-homogeneous singularity f_2 has a universal abelian cover with splice type equations of the form

$$\begin{aligned} z_1^{10} &= a z_2^3 + b z_3^3 \\ z_4^2 &= c z_2^3 + d z_3^3 \end{aligned}$$

with diagonal action of a group of order 12, generated by $[-i, -1, -1, i]$ and $[-1, \varepsilon^2, \varepsilon, -1]$, where ε is a primitive third root of unity.

The second type of quartic curve with an E_6-singularity, the one with ordinary bitangent, gives a series of singularities with the same topology as the series f_k (see Prop. 1):

$$g_k\colon \ (x^2 - y^2)^2 - y^3 z - z^{4+k} \ .$$

But this time the fourfold cover is not a complete intersection. There is no easy recipe for the equations. We look at the cone over the singular quartic curve C. As a quartic curve it is canonically embedded. Let K be the canonical line bundle on C. There is a unique line bundle L of degree one on C with $4L = K$. We determine the ring $\bigoplus H^0(C, nL)$. We have $H^0(C, L) = 0$: an effective divisor of degree 1 would give an hyperflex. But $h^0(C, 2L) = 1$, the effective divisor of degree 2 being the points of tangency of the bitangent. By Riemann-Roch $h^0(C, 3L) = 1$. So we have to find the unique effective divisor D of degree 3, for which there is a cubic curve with fourfold contact. The cubic lies in the system of contact cubics, which we obtain (as in the classical case of smooth quartics [5, §17]) by writing the equation of the curve in the form $Q^2 - Tz$, where $Q = x^2 - y^2 - z(ax + by + cz)$ and $T = y^3 - 2(x^2 - y^2)(ax + by + cz) + z(ax + by + cz)^2$. As the curve C is invariant under the involution $x \mapsto -x$, the cubic (being unique) is also invariant, from which we conclude that $a = 0$. Using the parametrisation $(x, y, z) = (st^3, t^4, (s^2 - t^2)^2)$ we get $T = \left[t^6 - (s^2 - t^2)(bt^4 + c(s^2 - t^2)^2) \right]^2$. Writing out the condition that the term in brackets is the square of a cubic form and solving gives $b = 3$, $c = -4$. So

$$Q = x^2 - y^2 - z(3y - 4z) \ ,$$
$$T = y^3 - 2(x^2 - y^2)(3y - 4z) + z(3y - 4z)^2 \ .$$

The three points in D are $(0 : 1 : 1)$ and $(\pm 2\sqrt{6} : 4 : 1)$ or in terms of the parametrisation $(0 : 1)$ and $(\pm\sqrt{3} : \sqrt{2})$. The curve C is the boomerang shaped curve (with the E_6 singularity at the tip) in the next figure. To make the bitangent visible the coordinate transformation $z \mapsto z - \frac{1}{2}y$ was applied. The affine chart $z = 1$ in the new coordinates is shown. The conic Q intersects C in the points of tangency of the bitangent and is tangent in the three points of contact with the contact cubic T, whose real locus consists of a very small, hardly visible oval, and an odd branch with one point at infinity (an inflexion point); the picture shows two arcs of this branch.
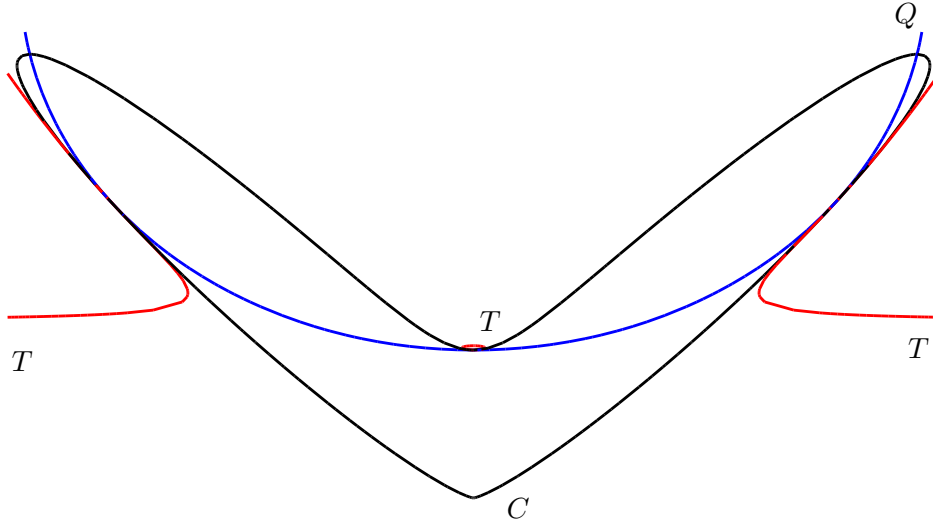
The picture was made with Richard Morris' program SINGSURF [8].

As $4D = 3K$ we have indeed that $L = K - D$ satisfies $4L = K$. We can describe $H^0(C, n(K - D))$ with polynomials of degree n, passing n times through D. In degree 2 we

have Q, in degree 3 T, in degree four Tx, Ty and $Tz = Q^2$. One section in degree 5 is TQ, the others are of the form TQ' with Q' a quadric through the three points in D. Alternatively we can write forms of degree n in s and t. The sections described so far generate the ring. As minimal set we choose the following generators:

$$
\begin{aligned}
\zeta: \quad & Q & = \quad & s^2 - t^2 \\
u: \quad & T & = \quad & s(2s^2 - 3t^2) \\
x: \quad & Tx & = \quad & st^3 \\
y: \quad & Ty & = \quad & t^4 \\
v: \quad & T(y - 4z)x & = \quad & -(2s^2 - t^2)t^3 \\
w: \quad & T(y - 4z)(y - z) & = \quad & s(2s^2 - t^2)(s^2 - 2t^2)
\end{aligned}
$$

One finds 9 equations between the 6 variables. They can be given in *rolling factors format* [23]. The first 6 equations are the 2×2 minors of the matrix

(3.a) $\qquad \begin{pmatrix} u & y - \zeta^2 & x & w \\ y - 4\zeta^2 & w & v & u^2 - 8\zeta^3 + 6y\zeta \end{pmatrix}$,

while the remaining 3 equations come from $u^2\zeta = Q(x, y, \zeta^2)$ by rolling factors. Specifically,

$$u^2\zeta - x^2 + (y - \zeta^2)(y + 4\zeta^2) \,,$$

(3.b) $\quad (y - 4\zeta^2)u\zeta - vx + w(y + 4\zeta^2) \,,$

$$(y - 4\zeta^2)^2\zeta - v^2 + (u^2 - 8\zeta^3 + 6y\zeta)(y + 4\zeta^2) \,.$$

By perturbing these equations we get again a series of singularities. We use a rolling factors deformation [23]: we deform the matrix and use the deformed matrix for rolling factors. As there is a recipe for the relations, basically in terms of the matrix, this yields a (flat) deformation of the singularity. The easiest way to perturb is to change the last entry of the matrix (3.a) into

(4) $u^2 - 8\zeta^3 + 6y\zeta + \varphi_k$

and correspondingly the term in the third additional equation of (3.b). We take $\varphi_{2k-1} = u\zeta^{1+k}$ and $\varphi_{2k} = \zeta^{3+k}$, because the singular locus of the quasi-cone $\oplus H^0(C, nL)$, $4L = K$, is given by $u^2 = 4\zeta^3$, $w = u\zeta$. As the ring $\oplus H^0(C, nL)$ is Gorenstein by construction, all the isolated singularities we obtain are Gorenstein. A weighted blow-up (with the weights of the non-isolated singularity) gives a partial resolution with the curve C as exceptional divisor and one singularity of suspension type $z^k = x^3 + y^4$.

By choosing an equivariant deformation for the $\mathbb{Z}_4$-action we ensure that it descends to the quotient. As quotient of the deformed singularities with φ_{4k} we obtain the Yomdin type series

$$\widetilde{g}_k: \ (x^2 - y^2)^2 - y^3z + (x^2 - y^2 - z(3y - 4z))z^{2+k} \,.$$

For $k = 1$ we have indeed a superisolated singularity. One computes that the Tyurina number of this singularity is 28, whereas $\tau(g_1) = 30$. So we did not compute the universal abelian covering of g_1, but of an analytically inequivalent singularity with the same degree 4 part. The general philosophy of superisolated singularities dictates that the covering of g_1 is a deformation of the same quasi-cone $\oplus H^0(C, nL)$. As it is not easy to give a direct formula, we start again with a rolling factors deformation, this time changing the $(2,1)$-entry of the matrix (3.a) into $y - 4\zeta^2 + 16\zeta^4 + 20y\zeta^2 + 21y^2 + 216\zeta^6$ and correspondingly the additional equations (3.b). We get a quotient of the form $(x^2 - y^2)^2 - v_1y^3z - v_2z^5$, where v_1 and v_2 are units, and an easy coordinate change brings us to g_1. The precise formulas for the units are rather complicated.

The computations so far can be used to give other examples. A related series is obtained by first taking the double cover. For a plane quartic with bitangent $z = 0$ with equation of the form $Q^2 - Tz$ the ring $\oplus H^0(C, n\Theta)$, where Θ is the line bundle belonging to the bitangent, has four generators, ζ in degree 1, with $\zeta^2 = z$, in degree 2 generators x and y and finally in degree 3 a generator $w = \sqrt{T}$. Equations are

$$\zeta w = Q(x, y, \zeta^2) ,$$
$$w^2 = T(x, y, \zeta^2) .$$

We obtain a series of singularities from our quartics with E_6 by changing the second equation into $w^2 = T(x, y, \zeta^2) + \psi_k$, the easiest being $\psi_k = \zeta^{6+k}$. We look at the first element of the series.

The family

$$\zeta w = x^2 - ty^2 ,$$
$$w^2 = y^3 + \zeta^7$$

is equisingular. Its minimal resolution has a rational curve with an E_6 of self-intersection -2 as exceptional divisor. The universal abelian cover is a double cover. For $t = 0$ it is the hypersurface singularity $\xi^4 - \eta^3 - \zeta^{14}$, while for $t \neq 0$ it has embedding dimension 6, being an appropriate deformation of our singular quasi-cone with equations (3.a) and (3.b); taking $\varphi_2 = \zeta^4$ in the perturbation (4) yields a quotient with a more complicated formula.

Finally, by looking at the lowest element of the series for the fourfold cover we get an example of a Gorenstein singularity with integral homology sphere link, which is not a complete intersection.

Proposition 2. *The Gorenstein singularity obtained by taking $\varphi_1 = u\zeta^2$ in (4), i.e., with equations*

$$\text{Rank} \begin{pmatrix} u & y - \zeta^2 & x & w \\ y - 4\zeta^2 & w & v & u^2 - 8\zeta^3 + 6y\zeta + u\zeta^2 \end{pmatrix} \leq 1 ,$$

$$u^2\zeta - x^2 + (y - \zeta^2)(y + 4\zeta^2) ,$$
$$(y - 4\zeta^2)u\zeta - vx + w(y + 4\zeta^2) ,$$
$$(y - 4\zeta^2)^2\zeta - v^2 + (u^2 - 8\zeta^3 + 6y\zeta + u\zeta^2)(y + 4\zeta^2) ,$$

has integral homology sphere link. Its minimal resolution has the rational curve with an E_6-singularity as exceptional divisor with self-intersection -1. The resolution graph is the same as for the hypersurface singularity $\xi^4 - \eta^3 - \zeta^{13}$:
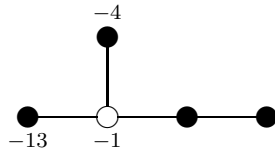
For the hypersurface singularity $p_g = 8$. There are several ways to compute this. One can determine the number of spectrum numbers less or equal to one, an easy computation for a Brieskorn polynomial — a lazy method is to use the computer algebra system SINGULAR [3] to find the spectrum. Or one can use the Laufer formula $\mu = 12p_g + K^2 + b_2(E) - b_1(E)$ on the minimal resolution. As explained at the beginning of the next section, the geometric genus can also be computed as $\sum_{n=0}^{4} h^0(C, nL)$ on the singular curve. For the non hypersurface singularity one finds in this way $p_g = 6$.

4.2. The three-cuspidal quartic. A quartic curve with three A_2-singularities has a bitangent, and equations for the fourfold cover can be found in the same way as in the case of a curve with an E_6 singularity and an ordinary bitangent.

Let σ_1, σ_2 and σ_3 be the elementary symmetric functions of x, y and z. The 3-cuspidal curve has equation $\sigma_2^2 - 4\sigma_1\sigma_3 = x^2y^2 + x^2z^2 + y^2z^2 - 2x^2yz - 2xy^2z - 2xyz^2$. The bitangent is $\sigma_1 = x + y + z$. We have again $L = K - D$, with D a divisor of degree 3, consisting of the points $(1:4:4)$, $(4:1:4)$ and $(4:4:1)$. We write the curve in the form $Q^2 - 4T\sigma_1$, where the cubic form T cuts out $4D$. One computes

$$\begin{aligned}
Q &= & 27\sigma_2 - 8\sigma_1^2 &= & -8(x^2 + y^2 + z^2) + 11(xy + xz + yz)\,, \\
T &= & 729\sigma_3 - 108\sigma_1\sigma_2 + 16\sigma_1^3 &= & \\
& & \multicolumn{3}{l}{16(x^3 + y^3 + z^3) - 60(x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2) + 501xyz\,.}
\end{aligned}$$

The quadric Q is a section of $2L$ and the cubic T a section of $3L$. A basis of sections of $4L = 4K - 4D \equiv K$ consists of Tx, Ty and Tz; due to the relation $Q^2 = 4T\sigma_1$ we need only two of these as generators of the ring. The necessary choice breaks the symmetry. In degree five we have sections of the form TQ', where Q' is a quadric passing though the three points of the divisor D. We choose as generators two forms with reducible quadrics, consisting of the line through two of the points and a line though the third point. The equations are again in rolling factors format, with the transition from the first to the second row being multiplication with the equation of the chosen line. A suitable choice for the other line in a generator of degree 5 is the tangent in the third point. Therefore we take as generators $\frac{1}{2}Q$, T, Ty, Tz, $T(y - z)(5x - 4y - 4z)$ and $T(8x - y - z)(5x - 4y - 4z)$, which correspond to coordinates (s, u, y, z, v, w).

We get a series of isolated singularities by a rolling factors deformation. With the given generators we form the matrix

$$(5.\text{a}) \quad \begin{pmatrix} u & y - z & 8s^2 - 9y - 9z & w \\ 5s^2 - 9y - 9z & v & w & u^2 - 46s^3 + 54ys + 54zs + \varphi_k \end{pmatrix},$$

while the last 3 equations come from $u^2s = Q(s^2 - y - z, y, z)$ by rolling factors. Specifically,

$$(5.\text{b}) \quad \begin{aligned}
&8u^2s + (8s^2 - 9y - 9z)(4s^2 - 9y - 9z) + 27(y - z)^2\,, \\
&8(5s^2 - 9y - 9z)us + w(4s^2 - 9y - 9z) + 27(y - z)v\,, \\
&8(5s^2 - 9y - 9z)^2s + (u^2 - 46s^3 + 54ys + 54zs + \varphi_k)(4s^2 - 9y - 9z) + 27v^2\,.
\end{aligned}$$

We take $\varphi_{2k-1} = us^{1+k}$ and $\varphi_{2k} = s^{3+k}$, while $\varphi_\infty = 0$ gives the undeformed non-isolated singularity. A partial resolution exists with the curve C as exceptional divisor and three singularities of suspension type $z^k = x^2 + y^3$.

With $k = 4$ we get the universal abelian cover of a superisolated singularity. One gets a nicer quotient by throwing in some factors 9: the above equations with $\varphi_4 = 729s^5$ give the universal abelian cover of the superisolated singularity $\sigma_2^2 - 4\sigma_1\sigma_3 - 2Q\sigma_1^3$, which has a 3-cuspidal rational curve with self-intersection -4 as exceptional divisor on the minimal resolution.

Proposition 3. *The Gorenstein singularity in $(\mathbb{C}^6, 0)$ with equations (5.a,b) with $\varphi_1 = us^2$ has integral homology sphere link. The exceptional divisor on the minimal resolution is the*

three-cuspidal rational curve with self-intersection -1. *The resolution graph and splice diagram are as follows. The semigroup condition is not satisfied.*
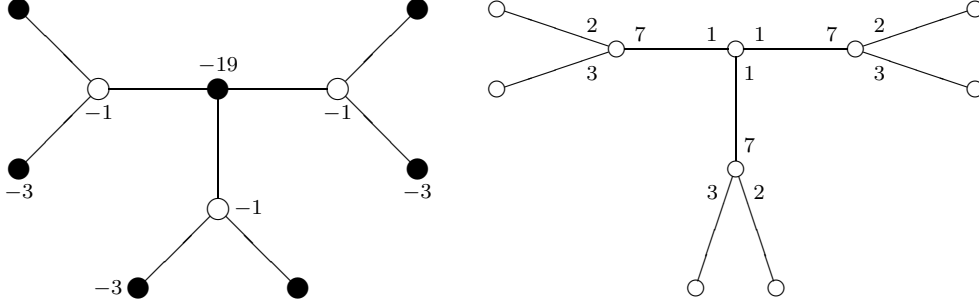
Indeed, the semigroup condition requires that a 1 adjacent to the central node is in the semigroup generated by 2 and 3, which is impossible.

This singularity is in fact a universal abelian cover of a hypersurface singularity, which will be considered in section 6.3.

4.3. The case A_6.

This is one of the examples made more explicit by Luengo-Velasco, Melle-Hernández and Némethi [7, 4.5]. A quartic curve with A_6 is unique up to projective equivalence. Its equation is $(zy - x^2)^2 - xy^3 = 0$. The associated superisolated singularity is different from the splice quotient associated to its resolution graph. In that case the corresponding curve of arithmetic genus 3 is the weighted complete intersection $yz = x^2$, $t^2 = xy^3$. This shows that the curve is hyperelliptic: the canonical linear system is not very ample. Seven Weierstraß points are concentrated in the singular point, while $P \colon (x : y : z) = (0 : 1 : 0)$ is an ordinary Weierstraß point. One has $4P = K$. The ring $\oplus H^0(C, nP)$ is simply given by $\{u^7 = w^2\} \subset \mathbb{C}^3$.

The case $A_4 + A_2$ is similar, in that the splice quotient comes from an hyperelliptic curve, while the plane quartic is unique with equation $(xy - z^2)^2 - yx^3 = 0$.

The plane quartic with an A_6 is a deformation of the hyperelliptic curve. As $2P$ is the g_2^1, which is an even theta-characteristic, the plane quartic with an A_6 has a unique ineffective theta-characteristic Θ, which is a line bundle. Then it is well known, going back to Hesse, that one can write the equation of the curve as linear symmetric determinant. It is possible to compute the matrix, but the easiest approach is to search for it in Wall's classification of nets of quadrics [25]. After some elementary operations we find the matrix

$$M = \begin{pmatrix} y & x & z & 0 \\ x & 4z & 0 & -y \\ z & 0 & x & x \\ 0 & -y & x & 0 \end{pmatrix} .$$

We get a series of singularities by changing the last entry of the matrix. In particular, replacing the 0 on the diagonal by z^2 leads to the superisolated singularity

$$(zy - x^2)^2 - xy^3 + z^2(4yzx - x^3 - 4z^3) \ .$$

The matrix M not only gives the curve C, as its determinant, but also the embedding by $K + \Theta = 3\Theta$ in $\mathbb{P}^3$: M is the matrix of a net of quadrics in $\mathbb{P}^3$ and the curve is the Steiner curve of this net, the locus of vertices of singular quadrics. The matrix also determines the equations (and in fact the whole resolution of the ideal) of the ring $\oplus H^0(C, n\Theta)$. This ring has generators x, y and z in degree 2 and four generators $v_0, \ldots v_3$ in degree 3. There are 14 equations, which can be written succinctly as matrix equations

$$Mv = 0 \ , \qquad vv^t = \bigwedge^3 M \ ,$$

where v is the column vector $(v_0, v_1, v_2, v_3)^t$. The same equations for the perturbed matrix define a double cover of the superisolated singularity. It has the same topology as the hypersurface singularity $v^{10} - vu^7 + t^2 = 0$, but different p_g. The link is a rational homology sphere. The resolution graph is
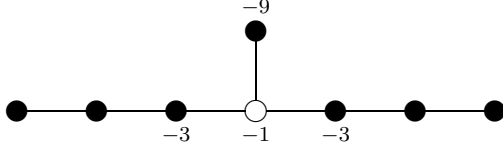
For the fourfold cover we have to find the fourth root L of K. We look again for an effective divisor D of degree 3 such that $4D = 3K$ and $L = K - D$. We start by computing the sections of $3\Theta = 6L$ in terms of the parametrisation $(x : y : z) = (s^2t^2 : t^4 : s^4 + st^3)$. Then we determine which section is a perfect square. We find the divisor given by $4s^3 + 3t^3$. We now use the following general procedure to compute the ring $\bigoplus H^0(C, nL)$ in terms of the parametrisation. Sections of $5L$ are found by finding the sections of $2K = 8L$, which are divisible by $4s^3 + 3t^3$. The sections of $7L$ are found from the sections of $10L$. Knowing the sections in sufficiently many degrees we can figure out the generators of the ring. There are 12 generators:

$$
\begin{aligned}
&\text{degree 3:} &&4s^3 + 3t^3, \\
&\text{degree 4:} &&s^2t^2, \quad t^4, \quad s^4 + st^3, \\
(6)\quad &\text{degree 5:} &&st^4, \quad 4s^3t^2 + t^5, \quad 4s^5 + 5s^2t^3, \\
&\text{degree 6:} &&t^6, \quad s^2t^4, \quad 2s^4t^2 + st^5, \\
&\text{degree 7:} &&t^7, \quad st^6.
\end{aligned}
$$

A computation shows that there are 54 equations between the generators. To describe their structure it is better to view the ring $R = \bigoplus H^0(C, nL)$ as module over $R_0 = \bigoplus H^0(C, nK) = \mathbb{C}[x, y, z]/((zy - x^2)^2 - xy^3)$. We have $R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$ with $R_i = \bigoplus_n H^0(C, (4n+i)L)$. We first look at R_3. As the section $(4s^3 + 3t^3)^2$ of $6L$ is a linear combination of the v_i, we can arrange that its square is a principal minor of a matrix defining the curve C. We change the matrix M into

$$
(7)\qquad M = \frac{1}{2}\begin{pmatrix} 0 & y & -x & 0 \\ y & -32z & -9y & 8x \\ -x & -9y & 10x & 8z \\ 0 & 8x & 8z & -8y \end{pmatrix}.
$$

The first principal minor is

$$
W = -80x^3 + 81y^3 + 176xyz + 256z^3 .
$$

This form cuts out $4D$ on C. We introduce a dummy variable $\sigma = 1/\sqrt[4]{W}$ of weight -3 (the variables x, y and z having weight 4), so satisfying $\sigma^4 W = 1$. Then $w = \sigma^3 W$ has indeed divisor D. We compute the ideal of $3D$ in the homogeneous coordinate ring of C; it is generated by W and

$$
-28x^3y + 27y^4 + 44xy^2z - 64x^2z^2 + 64yz^3 ,
$$
$$
x^2y^2 - 16x^3z + 27y^3z + 80xyz^2 .
$$

We obtain variables r_1 and r_2 of degree 7, having the same divisors as the generators of degree 7 in (6), by multiplying these expressions with σ^3. The generators of R_2 are (w^2, v_1, v_2, v_3). We obtain them from the matrix M by $v_i = \sigma^2(\bigwedge^3 M)_{0i}$; the matrix is chosen in such a way that these sections correspond exactly to the generators in degree 6, given in (6) in terms of the parametrisation. For R_1 we look at sections of $5(K - D) \equiv 2K - D$. The linear system of

quadrics through the three points of D can be computed as the radical of the ideal generated by the four minors of M of the first row. We find the ideal

$$(3y^2 + 16xz, xy - 16z^2, x^2 + 3yz) \; .$$

Multiplying these generators by σ yields the variables u_1, u_2 and u_3. The dummy variable σ gives us the ring structure. By eliminating σ we find the equations.

The matrix M plays again an important role. Let $v = (w^2, v_1, v_2, v_3)^t$. Then

$$(8.a) \quad Mv = 0 \;, \qquad vv^t = {\textstyle\bigwedge}^3 M \;,$$

but some equations are consequences of others. We get also equations from the syzygies of the ideal $(3y^2 + 16xz, xy - 16z^2, x^2 + 3yz)$ giving (u_1, u_2, u_3) and from those of the ideal leading to (r_1, r_2, w). By a suitable choice of generators the same matrix can be used for both ideals (or rather the matrix and its transpose). Let r be the vector $(r_1, r_2, w)^t$ and $u = (u_1, u_2, u_3)^t$ and consider the matrix

$$N = \frac{1}{2} \begin{pmatrix} -x & 4z & -xy \\ 3y & 4x & -y^2 \\ 16z & -4y & 4x^2 - 4yz \end{pmatrix} \; .$$

Then

$$(8.b) \quad u^t N = 0 \;, \qquad Nr = 0 \;, \qquad ur^t + {\textstyle\bigwedge}^2 N = 0 \;.$$

The last expression includes for example the equation $u_1 w = 3y^2 + 16xz$, which obviously holds by our definition of the generators. The remaining equations concern the rewriting of monomials and they are in a certain sense a consequence of the given ones. E.g., the equation $w^3 - 27yu_1 + 16zu_2 + 80xu_3$ can be obtained because we know how to express w^4 and $u_i w$ in terms of x, y and z. But basically the equation boils down to expressing the cubic $W = -80x^3 + 81y^3 + 176xyz + 256z^3$, passing through D, in the generators of the ideal of D. The space defined by the matrix equations (8.a,b) alone has other components, but they all lie in the hyperplane $\{w = 0\}$. We can find the ideal of the singularity by saturating with respect to the variable w.

We obtain a superisolated singularity as determinant of a perturbation of the matrix M: we change the upper left entry of (7) into $8z^2$. To find its universal abelian cover we note that the intersection of the singularity with the cubic cone $W = 0$ still consists of three lines, counted with multiplicity four. We can therefore compute with the same forms, but now over the local ring of the singularity. In this instance we succeed in determining generators of the ring. That we now have a deformation of the non-isolated singularity follows because we only change the ring structure on the same underlying $\mathbb{C}$-module. Again we find equations by eliminating the dummy variable σ. As a consequence of our set-up the perturbed matrix M enters, but also N changes, as its determinant is a multiple of the defining equation of the hypersurface singularity.

Proposition 4. *The superisolated singularity*

$$(yz - x^2)^2 - xy^3 + (-80x^3 + 81y^3 + 176xyz + 256z^3)z^2 \;,$$

with a rational curve with an A_6-singularity as exceptional divisor, has as universal abelian cover a singularity of embedding dimension 12.

The universal abelian cover can be given by the following matrix equations and 30 additional equations:

$$Mv = 0 \;, \qquad vv^t = {\textstyle\bigwedge}^3 M \;, \qquad u^t N = 0 \;, \qquad Nr = 0 \;, \qquad ur^t + {\textstyle\bigwedge}^2 N = 0 \;.$$

Here $v = (w^2, v_1, v_2, v_3)^t$, $r = (r_1, r_2, w)^t$, $u = (u_1, u_2, u_3)^t$ are vectors involving nine of the variables and the matrices M and N depend on x, y and z:

$$M = \frac{1}{2} \begin{pmatrix} 8z^2 & y & -x & 0 \\ y & -32z & -9y & 8x \\ -x & -9y & 10x & 8z \\ 0 & 8x & 8z & -8y \end{pmatrix}$$

and

$$N = \frac{1}{2} \begin{pmatrix} -x & 4z & -xy + 108yz^2 \\ 3y & 4x & -y^2 - 64z^3 \\ 16z & -4y & 4x^2 - 4yz - 320xz^2 \end{pmatrix}.$$

The additional equations can be obtained by saturating with respect to the variable w. The equation of the superisolated singularity is $\frac{1}{4} \det M = -\frac{1}{2} \det N$.

5. Degree five

The first counterexamples to the conjectured formula for p_g of Némethi and Nicolaescu are superisolated singularities with $d = 5$ [7, 4.1]. The main result of this section is that a splice quotient with the same resolution graph, if existing, has the predicted p_g.

We first explain how to compute the geometric genus. Consider more generally a Gorenstein singularity (X, p) such that the exceptional divisor on the minimal resolution is an irreducible curve C with arithmetical genus $p_a = g$. The Gorenstein condition implies that the dual of the normal bundle of C is a line bundle L with $dL = K_C$ for some integer d, where K_C denotes the canonical sheaf on C, so $\deg L = \frac{2g-2}{d}$. For the canonical cycle on the minimal resolution we have $K = -(d+1)C$.

We look at the exact sequences

$$0 \longrightarrow \mathcal{O}(-(n+1)C) \longrightarrow \mathcal{O}(-nC) \longrightarrow \mathcal{O}_C(-nC) \longrightarrow 0$$

on the minimal resolution. An upper bound for p_g is $\sum_{n=0}^{d} h^1(\mathcal{O}_C(-nC)) = \sum_{n=0}^{d} h^1(C, nL)$, which by duality is equal to $\sum_{n=0}^{d} h^0(C, nL)$. We have equality if the maps $H^0(\mathcal{O}(-nC)) \to H^0(\mathcal{O}_C(-nC))$ are surjective. This is the case if the associated graded ring of the filtration defined by the vanishing order along C is the ring $\bigoplus H^0(C, nL)$. This is true by construction for the singularities obtained by deforming this ring. Splice quotients with the same graph can also be interpreted in this way. Okuma has given a formula for p_g for all splice quotients [18]. In our situation, if C is a rational cuspidal curve of arithmetic genus 6, and H is the unique line bundle of degree 5 with $2H = K_C$, then $p_g = 7 + h^0(C, H)$.

If the irreducible exceptional curve C with $C^2 = -d$ has only one singularity, with only one Puiseux pair (p, q), then a splice type equation is $x^p - y^q + z^{d+pq}$ with action $\frac{1}{d}(q, p, 1)$. The abstract curve C is given by $x^p = y^q$, where this equation is considered on a suitable scroll, such that its only singular point is at the origin of the affine (x, y)-chart. If there are two singular points, with Puiseux pairs (p, q) and (r, s) respectively, then one can again write splice type equations, and the abstract curve is given by $x^p = y^q$, $z^r = w^s$. If there are three singular points, the semigroup condition is no longer satisfied, and one cannot write splice type equations, [7, 4.3]. Indeed, the strict transform of the singular curve now gives a node in the splice diagram, and the edge weights next to it are all equal to 1, as in the splice diagram in Proposition 3.

We call an irreducible, locally plane Gorenstein curve hyperelliptic if the canonical system defines a $2:1$ cover of $\mathbb{P}^1$. The explicit description above of the curve C on the minimal resolution proves the following Proposition.

Proposition 5. *Let the exceptional divisor of a Gorenstein splice quotient be an irreducible rational cuspidal curve C with only singularities of type A_k. Then the curve C is hyperelliptic.*

In an A_{2k}-singularity $2k+1$ Weierstraß points, i.e., ramification points of the double cover of $\mathbb{P}^1$, are absorbed. The equations for C should be compared with the equations for a Gorenstein quasi-cone $X(C, L)$, where C is a smooth hyperelliptic curve and L is a line bundle of degree 1 with $2L$ the g_2^1. If $L = \mathcal{O}_C(P)$ with P a Weierstraß point, then we get an equation of type $x^2 + y^{2g+1} + z^{4g+2}$. If $L \sim P_1 + \cdots + P_{2k+1} - kg_2^1$, with the P_i Weierstraß points, then we get a complete intersection with equations of the type $x^2 = f_{2k+1}(y, w)$, $z^2 = g_{2g-2k+1}(y, w)$.

We now specialise to the case $d = 5$.

Proposition 6. *Let (X, p) be a splice quotient with exceptional divisor a rational cuspidal curve C with $p_a(C) = 6$ and $C^2 = -5$. Then $p_g(X)$ has the value predicted by the conjecture of Némethi and Nicolaescu.*

Such a singularity exists for every combination of at most two (higher) cusps. The following table gives the values in question.

singularities	p_g	singularities	p_g	singularities	p_g
W_{12}	10	$2E_6$	7	$A_{10} + A_2$	8
E_{12}	9	$E_8 + A_4$	10	$A_8 + A_4$	8
A_{12}	10	$E_6 + A_6$	8	$2A_6$	7

The predicted value for the cases realisable as plane quintic is taken from [7, 4.1], the other ones are calculated with the formulas in [7, 2.3]. The geometric genus of the splice quotients is easily computed. As $p_g = 7 + h^0(C, 5L)$, $\deg L = 1$, one only needs to calculate $h^0(C, 5L)$.

Below we comment on several cases. We take the opportunity to give other singularities with the same resolution graph, not necessarily superisolated singularities. For some singularities we have determined the universal abelian cover. To this end we have computed generators of rings of the type $\bigoplus H^0(C, nL)$. In most cases there are many equations between these generators, without apparent structure. It did not seem worthwhile to compute the corresponding series, or an explicit example of a universal abelian cover of a superisolated singularity.

5.1. **A_{12}.** The splice quotient has a hyperelliptic exceptional divisor. Its equations are described in [7, 4.6]. The superisolated singularity comes from the curve

$$C: \quad z(yz - x^2)^2 + 2xy^2(yz - x^2) + y^5 \ .$$

Let H be the g_5^2 on C and L the unique line bundle with $5L = H$. We determine generators for the ring $\bigoplus H^0(C, nL)$ in terms of the parametrisation of C. We start by determining all divisors D of degree 6 such that $5D$ is cut out by a plane sextic. We find a unique, reduced divisor, so $\dim H^0(C, 6L) = 1$, and $H^0(C, L) = H^0(C, 2L) = H^0(C, 3L) = 0$, as the divisor is not a multiple. By Riemann-Roch $H^0(C, 4L) = 0$, and we find $\dim H^0(C, nL)$ for all other n. The following table gives the generators of the ring.

degree 5 : $\quad t^2s^3 + t^5,\ st^4,\ s^5 + 2s^2t^3,$

degree 6 : $\quad s^6 + \frac{12}{5}s^3t^3 + \frac{4}{75}t^6,$

degree 7 : $\quad s^3t^4 + \frac{4}{5}t^7,\ s^7 + \frac{14}{5}s^4t^3 - \frac{28}{25}st^6,$

degree 8 : $\quad s^4t^4 + \frac{6}{5}st^7,\ s^6t^2 + \frac{11}{5}s^3t^5 + \frac{17}{25}t^8,\ s^8 + \frac{16}{5}s^5t^3 + \frac{112}{25}s^2t^6,$

degree 9 : $\quad s^3t^6 + \frac{3}{5}t^9,\ s^7t^2 + \frac{13}{5}s^4t^5 + \frac{28}{25}st^8,\ s^9 + \frac{18}{5}s^6t^3 + \frac{88}{25}s^3t^6,\ s^5t^4 + \frac{8}{5}s^2t^7,$

degree 11: $\quad st^{10},\ s^3t^8 + \frac{2}{5}t^{11},\ s^5t^6 + \frac{7}{5}s^2t^9.$

We spare the reader the 104 equations between these generators.

The general curve of arithmetical genus 6 with an A_{12}-singularity does not have an embedding into the plane. As example we look at a trigonal curve. Consider the curve

$$C: \quad (w - a^2)^2 - 2(w - a^2)aw^2 + a^4 w^3$$

on $\mathbb{P}^1 \times \mathbb{P}^1$ with parametrisation $w = t^4/(1 + 2t^3)$, $a = t^2/(1 + t^3)$. Its canonical embedding is $(1 : a : a^2 : w : wa : wa^2)$. After homogenising we have indeed 6 forms in s and t of degree 10. Let H be the unique line bundle of degree 5 with $2H = K_C$. It has a unique section, given by st^4. The ring $\bigoplus H^0(C, nH)$ has generators in degree 1, 2 and 3. The corresponding isolated singularity has $p_g = 8$. For the ring $\bigoplus H^0(C, nL)$, with $\deg L = 1$, $5L = H$, one finds that st^4 is the generator of lowest degree, so one needs generators in all degrees from 5 to 13.

5.2. $\boldsymbol{W_{12}}$.
In this case the curve singularity has equation $x^5 - y^4 - ay^2 x^3$. The results here are similar to the case of an E_6-singularity on a quartic curve.

The quasi-homogeneous case occurs as splice quotient. The equation is $\xi^5 - \eta^4 - \zeta^{26}$, and invariants for the $\mathbb{Z}_5$-action are $x = \xi\zeta$, $y = \eta$ and $z = \zeta^5$, which gives the superisolated singularity $x^5 - y^4 z - z^6$.

The non quasi-homogeneous form occurs for the superisolated singularity $x^5 - y^4 z - y^2 x^3 - z^6$. We parameterise the exceptional curve, and compute the divisors D of degree 6, such that $5D$ is cut out by a sextic. This time there is a pencil, so the ring $H^0(C, nL)$, where $\deg L = 1$, has one generator in degree 4.

Both singularities occur also on projectively non equivalent curves, $x^5 - y^4 z - y^3 x^2$ and $x^5 - y^4 z - y^2 x^3 - y^3 x^2$. In both cases the ring $\bigoplus H^0(C, nL)$ has no generators of degree less than 5, and the number of generators is equal to the number found for the plane curve with an A_{12}.

5.3. $\boldsymbol{E_{12}}$.
There is no plane quintic with a singularity of this type, but we can form a splice quotient for the graph. It is the quotient of $\xi^3 + \eta^7 + \zeta^{26}$ under the action $\frac{1}{5}(2, 3, 1)$.

This gives us a clue for writing down rational curves with an E_{12}-singularity. The canonical embedding of the trigonal rational curve $\xi^3 + \eta^7 + a\xi\eta^5$ is given by forms of weighted degree at most 10, where we give the variables (ξ, η) the weights $(7, 3)$. We set $y_i = \eta^i$, $x_i = \xi\eta^i$. The equations are in rolling factors format:

$$\text{Rank} \begin{pmatrix} y_0 & y_1 & y_2 & x_0 \\ y_1 & y_2 & y_3 & x_1 \end{pmatrix} \leq 1 \, ,$$

$$x_0^3 + y_3^2 y_1 + ax_0 y_2 y_3 \, ,$$

$$x_0^2 x_1 + y_3^2 y_2 + ax_1 y_2 y_3 \, ,$$

$$x_0 x_1^2 + y_3^3 + ax_1 y_3^2 \, .$$

For $a = 0$ we have the exceptional curve of the splice type singularity, but for $a \neq 0$ the trigonal curve is not invariant under the $\mathbb{Z}_5$-action. In this case we find that the lowest degree generator of the ring $\bigoplus H^0(C, nL)$, $\deg L = 1$, has degree 6. This means that the ring of $H = 5L$ has no generator in degree 1, but 6 generators in degree 2 and 10 in degree 3. The corresponding isolated singularity has $p_g = 7$.

5.4. $\boldsymbol{E_8 + A_4}$.
The splice quotient is a superisolated singularity with homogeneous part $x^3 z^2 - y^5$. One has also the curve $x(xz - y^2)^2 - y^5$, for which the ring $\bigoplus H^0(C, nL)$ has no generators of degree less than 5.

5.5. $\boldsymbol{E_6 + A_6}$.
The splice quotient has high embedding dimension. The corresponding singular curve is tetragonal. The plane curve $x(xz + y^2)^2 + 2y^3(xz + y^2) - y^4 z$ leads to $\bigoplus H^0(C, nL)$ without generators of degree less than 5.

5.6. **$2E_6$.** The exceptional curve of the splice quotient is a trigonal curve with bihomogeneous equation $x_1^3 y_2^4 - x_2^3 y_1^4$ on $\mathbb{P}^1 \times \mathbb{P}^1$. As the ring $\bigoplus H^0(C, nL)$ has generators in degree 3 and 4, there are no sections of $5L$, so the splice quotient with $E^2 = -5$ on the minimal resolution has maximal ideal cycle $2E$ on the minimal resolution.

The most general canonical curve of genus 6 is the complete intersection of a quintic Del Pezzo surface with a quadric. A plane model for such a curve is a sextic with 4 double points. An example with 2 E_6-singularities is

$$(xz + yz - 4xy)^2 z^2 - 4x^3 y^3 \ ,$$

with a double point in $(1 : 1 : 1)$ and three infinitely near double points at $(0 : 0 : 1)$. This curve is still rather special, which manifests itself in the fact that the ring $\bigoplus H^0(C, nL)$ has one generator in degree 4.

5.7. **Rational plane quintics with three or four cusps.** The superisolated singularities with these exceptional curves have all $p_g = 10$, whereas the Némethi-Nicolaescu formula gives a lower value, see the table in [7]. In the case $E_6 + A_4 + A_2$ the predicted value 8 is realised by a general curve of arithmetical genus 6 with these singularities. In the two remaining cases the predicted value is less than 6. This can never be realised by a normal surface singularity, as one plus the arithmetical genus of the exceptional divisor gives a lower bound for the geometric genus of the singularity.

6. Generalised superisolated singularities

The decisive property of the superisolated singularities studied above is that the exceptional locus of the minimal resolution consists of one irreducible curve. With the cone over this curve comes a whole Yomdin series of singularities, whose lowest element is a superisolated singularity.

A natural generalisation is to look at the series of weighted homogeneous curves. Let L be an ample line bundle on a singular curve. The quasi-cone $X(C, L)$ is the singularity with local ring $\bigoplus H^0(C, nL)$. If this non-isolated singularity is a hypersurface singularity, we obtain a Yomdin series by adding high powers of a linear form. If the singular locus has itself singular branches, this series can be refined. In general, in the non hypersurface case, there will also be Yomdin type series, as in the examples for degree four, but there is no easy general formula. To resolve such singularities one starts with a weighted blow-up. The singular curve is then the exceptional set. If this blow-up resolves the singularity, we speak of a generalised superisolated singularity. To obtain a generalised superisolated singularity with rational homology sphere link from an irreducible singular curve the curve has to be locally irreducible.

6.1. **Rational curves with an E_8-singularity.** A rational curve with an E_8-singularity has arithmetical genus 4. In its canonical embedding it is a complete intersection of a quadric and a cubic. A corresponding superisolated complete intersection splice quotient is the quotient of $x^3 + y^5 + z^{21}$. With variables $d = xz$, $c = z^3 y$, $b = z^6$ and $a = y^2$ we get the equations $c^2 - ab = d^3 + a^2 c + b^4$. The line bundle $L = \frac{1}{2}K$ gives rise to a weighted superisolated singularity $d^3 + \alpha^5 \beta + \beta^7$, which is the quotient of $x^3 + y^5 + z^{18}$ by a group of order 3.

For both singularities we have a deformation of positive weight, such that the universal abelian cover is not a deformation of the splice type equation. We consider $c^2 - ab = d^3 + a^2 c + a^2 d + b^4$ and $d^3 + \alpha^5 \beta + \alpha^4 d + \beta^7$. To study coverings we parameterise the rational curve $c^2 - ab = d^3 + a^2 c + a^2 d = 0$ by

$$(a, b, c, d) = (s^6, (s^2 + t^2)^2 t^2, -(s^2 + t^2)ts^3, ts^5) \ .$$

Let $L = \frac{1}{6}K$. Generators of the ring $\oplus H^0(C, nL)$ are

degree 3: s^3, $s^2t + t^3$,
degree 4: $2s^4 + 12s^2t^2 + 9t^4$,
degree 5: $5s^4t + 15s^2t^3 + 9t^5$, $s^5 + 3s^3t^2$,
degree 6: s^5t,
degree 7: s^6t, $s^7 + 3s^5t^2$.

There are 20 equations, which we suppress. In particular we obtain that the singularity $d^3 + \alpha^5\beta + \beta^7 + t\alpha^4 d$ is a splice quotient only for $t = 0$.

The most general rational curve with an E_8-singularity lies on a smooth quadric. Equations are $ad - bc = (a+d)^3 + c^2 a + \lambda c^2 d = 0$ with modulus λ. Let again $L = \frac{1}{6}K$. A computation with the special value $\lambda = 0$ shows that $H^0(C, 3L) = 0$. The ring $\oplus H^0(C, nL)$ has 23 generators, of degrees 4 to 9. The ring of the double cover corresponding to $3L = \frac{1}{2}K$ has 4 generators in degree 2 and 6 generators in degree 3. The ideal is generated by 35 equations.

6.2. Canonical curves with g cusps.

A parametrisation of a curve with only ordinary cusps is easily given [22]. Let $\varphi_1, \ldots, \varphi_g$ be distinct linear forms in s and t. The map $(\varphi_1^2 : \cdots : \varphi_g^2)$ embeds $\mathbb{P}^1$ as conic in $\mathbb{P}^{g-1}$, which is tangent to the coordinate hyperplanes. The reciprocal transformation $(z_1 : \cdots : z_g) \mapsto (1/z_1 : \cdots : 1/z_g)$ sends it to a g-cuspidal curve of degree $2g - 2$.

We consider in particular the case $g = 4$. The existence of sections of roots of the line bundle K_C depends on the modulus of the curve. First let the cross ratio of the four cusps on $\mathbb{P}^1$ be harmonic. We parameterise the canonical curve as follows:

$$
\begin{aligned}
a &= (t+s)^2(t-is)^2(t+is)^2, \\
b &= -(t-s)^2(t-is)^2(t+is)^2, \\
c &= i(t-s)^2(t+s)^2(t+is)^2, \\
d &= -i(t-s)^2(t+s)^2(t-is)^2,
\end{aligned}
$$

and get equations $4ab - (a+b)(c+d) + 4cd = abc + abd + acd + bcd = 0$. The ring $\oplus H^0(C, nL)$, where $L = \frac{1}{6}K$, has 8 generators, and we get again 20 equations. The even subring $\oplus H^0(C, 2nL)$ gives a complete intersection. With $x = st$, $y = s^4 + t^4$, $z_1 = (s^2 + t^2)^3$ and $z_2 = (s^2 - t^2)^3$ we get the equations $z_1^2 - (y + 2x^2)^3 = z_2^2 - (y - 2x^2)^3 = 0$. A weighted superisolated singularity with exceptional curve a 4-cuspidal rational curve of self-intersection -2 is then given by the equations $z_1^2 - (y + 2x^2)^3 + x^7 = z_2^2 - (y - 2x^2)^3 + x^7 = 0$. It has Milnor number 65 and $p_g = 8$.

For a general cross ratio there are no generators of low degree, and we get again 23 generators for the universal abelian cover. The ring $\oplus H^0(C, 2nL)$ is also not a complete intersection.

6.3. A further generalisation.

The methods of this paper can be applied in even more general situations. We can study rings of the type $\bigoplus H^0(C, nL)$ also for $\mathbb{Q}$-divisors on singular curves. As example we look at the following graph of [14, Examples 2], which is an example where the semigroup condition is not satisfied for the splice diagram.
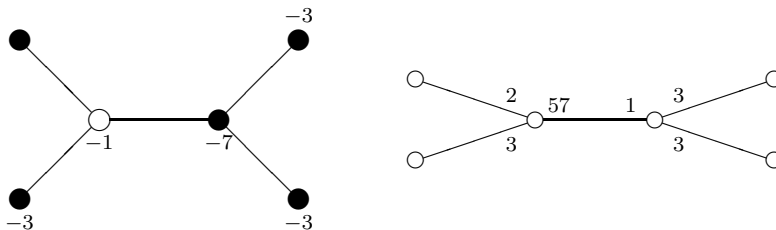
A Gorenstein singularity with this resolution graph exists. To find it we first look at the minimal resolution graph (as described in the first section):

$$\overset{-3}{\bullet}\!\!\!-\!\!\!-\!\!\!-\underset{[1]}{\overset{-1}{\circ}}\!\!\!-\!\!\!-\!\!\!-\overset{-3}{\bullet}$$
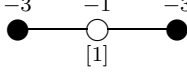
The weight below the central vertex is the arithmetic genus of the central cuspidal curve E_0. This is a star-shaped graph. Consider the $\mathbb{Q}$-divisor $D = P - \frac{1}{3}Q_1 - \frac{1}{3}Q_2$ on E_0, where Q_1 and Q_2 are the intersection points with the other components E_1 and E_2, and P is determined by the normal bundle. We have $K = -5E_0 - 2E_1 - 2E_2$, so the Gorenstein condition is $4P - 2Q_1 - 2Q_2 = 0$ in $\mathrm{Pic}(E_0)$. We satisfy it by taking $2P = Q_1 + Q_2$. We compute the graded ring $\bigoplus H^0(E_0, \lfloor nD \rfloor)$ for E_0 a cuspidal rational curve. We find the hypersurface singularity $z^2 = (x^2 - y^3)^3$. An isolated singularity with the original resolution graph is obtained by adding generic terms of higher weight. An example is $z^2 = (x^2 - y^3)^3 + xy^8$. The branch curve is irreducible and has Puiseux pairs $(3, 2)$ and $(10, 3)$.

The topology of the universal abelian cover is easily computed. The minimal resolution has as exceptional divisor a 3-cuspidal rational curve of self-intersection -1. Such a singularity was studied above in section 4.2. We use the notation and equations introduced there. A suitable $\mathbb{Z}_3$-action on the 3-cuspidal curve C is a cyclic permutation of the cusps, i.e., a cyclic permutation of the coordinates on $\mathbb{P}^2$. The polynomials Q and T are invariant under all permutations. The polynomial $27(x^2y + y^2z + z^2x - x^2z - y^2x - z^2y)$ is a skew invariant, which gives a section ψ of $3K - D \equiv 9(K - D)$. The cyclic permutation induces an action on the coordinates (s, u, y, z, v, w) of the equations (5.a,b). As the variables s and u correspond to Q and T, they are invariant. Furthermore $y \mapsto z \mapsto s^2 - y - z$, and v, w and us are involved in more complicated formulas. The ring of invariants of the action on $\bigoplus H^0(C, nL)$ is generated by s, u and ψ, and there is one equation

$$27\psi^2 + (4s^3 + u^2)^3 = 0 \ .$$

An explicit isolated singularity with C as exceptional divisor with $C^2 = -1$ is given in Proposition 3. Its equations are written explicitly in the introduction. As they are obtained by the invariant perturbation $\varphi_1 = us^2$, the same $\mathbb{Z}_3$-action on the coordinates (s, u, y, z, v, w) acts on the singularity, and we find the quotient by expressing ψ^2 in the local ring of the singularity in terms of s and u. We get a hypersurface singularity, which is equisingular with $z^2 = (x^2 - y^3)^3 + xy^8$, but has a more complicated equation.

Proposition 7. *The isolated singularity of Proposition 3 is the universal abelian cover (of order 3) of the double point*

$$27\psi^2 + (4s^3 + u^2)^3 + 2u^5 s^2 - 20u^3 s^5 - 4us^8 + u^4 s^4 \ ,$$

whose branch curve is irreducible, with Puiseux pairs $(3, 2)$ and $(10, 3)$.

7. Discussion

Superisolated singularities allow us to import the theory of projective curves into the subject of surface singularities. Our singularities with rational homology sphere links do not behave as rational curves, but have the character of curves of their arithmetical genus. The conjectures of Neumann and Wahl ask for the existence of special divisors (those cut out on the complete intersection by the coordinate hyperplanes). For smooth curves this is a complicated problem; the only thing we know for sure is that those of low degree do not exist on curves with generic moduli.

The idea behind the Casson Invariant Conjecture of Neumann and Wahl is that the Milnor fibre of a complete intersection is a natural four-manifold attached to its boundary, the link of the singularity, whose signature in the integral homology sphere case computes the Casson invariant of the link exactly. Starting from a graph the construction of Neumann and Wahl is

certainly very natural. Our computations for graphs of superisolated singularities with $d = 5$ support the following conjecture.

Conjecture 1. *Among the $\mathbb{Q}$-Gorenstein singularities with rational homology link the splice quotients have geometric genus as predicted by the Seiberg-Witten Invariant Conjecture of Némethi and Nicolaescu.*

This conjecture has now been proved by Némethi and Okuma, first in the integral homology sphere case [11], where it reduces to the Casson Invariant Conjecture, and later in general. More generally, Braun and Némethi prove an equivariant version [2]. An important ingredient is Okuma's p_g-formula [18].

In all examples we computed, where the universal abelian cover is not of splice type, it is not a complete intersection. The same is true for our Gorenstein singularities with integral homology sphere link. The following conjecture of Neumann and Wahl is still open.

Conjecture 2. *A complete intersection surface singularity with integral homology sphere link has equations of splice type.*

On the other hand, for all graphs we studied we found a singularity which is not a complete intersection. In fact, we conjecture that this is the general behaviour. If the exceptional divisor is a reduced and irreducible curve, one takes this curve general in moduli (whatever this means, as it is no so clear whether a sensible moduli space exists for curves with given singularities). Of course this does not apply to rational and elliptic singularities. We also exclude the case that the curve is hyperelliptic.

Conjecture 3. *The 'general' $\mathbb{Q}$-Gorenstein singularity with given rational homology sphere link is not a splice quotient, if the fundamental cycle has arithmetic genus at least 3. Its universal abelian cover is not a complete intersection.*

<div align="center">REFERENCES</div>

[1] Enrique Artal Bartolo, Ignacio Luengo and Alejandro Melle Hernández, Superisolated Surface Singularities, in: Singularities and Computer Algebra, London Math. Soc. Lect. Note Ser., **324** (Cambridge University Press, 2006), pp. 13–39.

[2] Gábor Braun and András Némethi, Surgery formula for Seiberg–Witten invariants of negative definite plumbed 3-manifolds, `arXiv:0704.3145`.

[3] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 2.0.5, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern (2003), `http://www.singular.uni-kl.de`

[4] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics **52** (Springer, New York, 1977).

[5] Felix Klein, Zur Theorie der Abel'schen Functionen, Math. Ann. **36**, 1–83 (1890).

[6] Ignacio Luengo, The μ-constant stratum is not smooth, Invent. Math. **90**, 139–152 (1987).

[7] I. Luengo-Velasco, A. Melle-Hernández and A. Némethi, Links and analytic invariants of superisolated singularities, J. Algebraic Geom. **14**, 543–565 (2005).

[8] Richard Morris, SingSurf, A Program for calculating singular algebraic curves and surfaces, 1990–2003, `http://www.singsurf.org/singsurf/SingSurfPro.html`

[9] András Némethi, "Weakly" ellipic Gorenstein singularities of surfaces, Invent. Math. **137**, 145–167 (199).

[10] András Némethi and Liviu I. Nicolaescu, Seiberg-Witten invariants and surface singularities, Geom. Topol. **6**, 269–328 (2002).

[11] András Némethi and Tomohiro Okuma, On the Casson Invariant Conjecture of Neumann–Wahl, `arXiv:math.AG/0610465`

[12] Walter D. Neumann, Abelian covers of quasihomogeneous surface singularities, in: Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., **40** (Amer. Math. Soc., Providence, RI, 1983), pp. 233–243.

[13] Walter D. Neumann and Jonathan Wahl, Casson invariant of links of singularities, Comment. Math. Helv. **65**, 58–78 (1990).

[14] Walter D. Neumann and Jonathan Wahl, Universal abelian covers of surface singularities, in: Trends in singularities, Trends Math., (Birkhäuser, Basel, 2002), pp. 181–190.

[15] Walter D. Neumann and Jonathan Wahl, Complete intersection singularities of splice type as universal abelian covers, Geom. Topol. **9**, 699–755 (2005).

[16] Tomohiro Okuma, Universal abelian covers of rational surface singularities, J. Lond. Math. Soc., **70**, 307–324 (2004).

[17] Tomohiro Okuma, Numerical Gorenstein elliptic singularities, Math. Z., **249**, 31–62 (2005).

[18] Tomohiro Okuma, The geometric genus of splice-quotient singularities, `arXiv:math.AG/0610464`.

[19] H. Pinkham, Normal surface singularities with C^* action, Math. Ann. **227**, 183–193 (1977).

[20] Miles Reid, Surfaces with $p_g = 3, K^2 = 4$ according to E. Horikawa and D. Dicks, Text of a lecture, Univ. of Utah and Univ. of Tokyo 1989.

[21] George Salmon, A treatise on the higher plane curves: intended as a sequel to A treatise on conic sections, 3rd Ed. (Hodges, Foster, and Figgis, Dublin 1879).

[22] Jan Stevens, On the number of points determining a canonical curve, Indag. Math. **51**, 485–494 (1989).

[23] Jan Stevens, Rolling Factors Deformations and Extensions of Canonical Curves, Documenta Math. **6**, 185–226 (2001).

[24] Jonathan Wahl, Topology, geometry, and equations of normal surface singularities, in: Singularities and Computer Algebra, London Math. Soc. Lect. Note Ser., **324** (Cambridge University Press, 2006), pp. 351–371.

[25] C.T.C. Wall, Nets of quadrics, and theta-characteristics of singular curves, Philos. Trans. Roy. Soc. London Ser. A **289**, 229–269 (1978).

E-mail address: stevens@math.chalmers.se

Matematiska Vetenskaper, Göteborgs universitet, SE 412 96 Göteborg, Sweden.